Увод в теорията на операторни алгебри с приложение в математиката и физиката
Introduction to Theory of Operator Algebras with Applications to Mathematics and Physics

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Lecture 4 (does not use Lecture 3)

Gelfand Theory (теория на Гелфанд) : algebraic part.

1. Maximal ideals (максимални идеали) and characters (характери)

__Def.__ A left/right/two-sided ideal $\mathcal{J} \subseteq \mathcal{A}$ in an associative algebra $\mathcal{A}$ is called proper (собствен идеал) iff $\mathcal{J} \neq \mathcal{A}$.
A left/right/two-sided ideal $\mathcal{J} \subseteq \mathcal{A}$ is called maximal left/right/two-sided ideal iff it is proper and there is no proper left/right/two-sided ideal $\mathcal{J}' \subseteq \mathcal{A}$ s.t. $\mathcal{J} \subsetneq \mathcal{J}'$.

__Poposition 4.1__ Let $\mathcal{A}$ be a unital associative algebra. Then $\mathcal{A}$ is division algebra iff ($\Leftrightarrow$) $\forall$ proper left and $\forall$ proper right ideals are zero.

__Poof.__ $\Rightarrow$) Let $\mathcal{J}$ be left ideal (and similarly for right). If $a \in \mathcal{J}$ and $a \neq 0$ then $\exists a^{-1}$. $\Rightarrow 1 = a^{-1} \cdot a \in a^{-1} \cdot \mathcal{J} \subseteq \mathcal{J}$. $\Rightarrow c = c \cdot 1 \in c \cdot \mathcal{J} \subseteq \mathcal{J}$ $(\forall c \in \mathcal{A})$.
$\Rightarrow \mathcal{J} = \mathcal{A}$, i.e. $\mathcal{J}$ is not proper.

$\Leftarrow$) Assume $\forall$ proper left ideal is zero (and similarly for right).
Let $a \in \mathcal{A}$, $a \neq 0$. Then $\mathcal{J} = \mathcal{A} . a$ is a left ideal. $a = 1 . a \in \mathcal{J}$. $\Rightarrow \mathcal{J} \neq 0$.
$\Rightarrow \mathcal{J}$ is not proper, i.e. $\mathcal{J} = \mathcal{A}$. In particular $1 \in \mathcal{J}$, i.e. $\exists c \in \mathcal{A} : c \cdot a = 1$. $\square$

__Precaution__. Proposition 4.1 is not true for two-sided ideals unless the algebra $\mathcal{A}$ is commutative. Example: $\mathrm{Mat}(n, \mathbb{C})$.

__Corollary 4.2.__ Let $\mathcal{A}$ be a unital associative algebra and $\mathcal{J} \subseteq \mathcal{A}$ be a two-sided ideal. Then a $\mathcal{J}$ is a maximal left and maximal right ideal iff $\mathcal{A}/\mathcal{J}$ is a division algebra.

__Proof.__ Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ be the canonical projection.

Then: $\quad \mathcal{J} \subsetneq \mathcal{J}' \subsetneq \mathcal{A} \quad \Rightarrow \quad 0 \subsetneq \pi(\mathcal{J}') \subsetneq \mathcal{A}/\mathcal{J}$ $\left(\begin{matrix}\text{since} \\ \pi^{-1}(\pi(\mathcal{J}')) = \mathcal{J}'\end{matrix}\right)$

$\Updownarrow$ intermediate $\begin{matrix}\text{left} \\ \text{right}\end{matrix}$ ideals $\Updownarrow$

Conversely: $\quad 0 \subsetneq I \subsetneq \mathcal{A}/\mathcal{J} \quad \Rightarrow \quad \mathcal{J} \subsetneq \pi^{-1}(I) \subsetneq \mathcal{A}$ $\left(\begin{matrix}\text{since} \\ \pi(\pi^{-1}(I)) = I\end{matrix}\right)$. $\square$

There are two common cases when a division complex algebra coincides with the algebra of complex numbers:

a) the case of finitely generated complex algebras

b) the case of Banach algebras (Gelfand-Mazur Theorem / Гельфанд-Мазур from lecture 2)

Example The algebra of rational functions $\mathbb{C}(t)$ is a division algebra and $\mathbb{C}(t) \ncong \mathbb{C}$. But $\mathbb{C}(t)$ is not finitely generated: it has a continuum (континуум) set of generators $(t-\alpha)^{-1}$, $\alpha \in \mathbb{C}$.

Def. A character (характер) of a unital associative complex algebra $\mathcal{A}$ is a nonzero morphism $\chi: \mathcal{A} \to \mathbb{C}$ of unital algebras.

By Corollary 4.2, if $\mathcal{A}$ is commutative then $\operatorname{Ker} \chi$ is a maximal (two-sided) ideal but the converse is not always true.

Theorem 4.3 Let $\mathcal{A}$ be a unital commutative Banach algebra.

(a) Every maximal (left = right = two-sided) ideal is closed.

(b) The correspondence: $\chi \mapsto \operatorname{Ker} \chi$ is a one-to-one correspondence between characters and maximal ideals.

(c) Every character χ is bounded and $\|\chi\| = 1$.

Observation 4.4 Let N.I.E.$(\mathcal{A})$ be the subset of non-invertible elements in a unital associative algebra $\mathcal{A}$.

(a) Every left (or right) proper ideal $\mathcal{J} \subseteq$ N.I.E.$(\mathcal{A})$.

(b) If $\mathcal{A}$ is a Banach algebra then N.I.E$(\mathcal{A})$ is a closed subset in $\mathcal{A}$.

Proof. (a) If $\exists a^{-1}$, $a^{-1} \in \mathcal{J}$ - left ideal, then: $1 = a \cdot a^{-1} \in \mathcal{J}$. $\Rightarrow \mathcal{J} = \mathcal{A}$.

(b) Why $\mathcal{A} \setminus$ N.I.E.$(\mathcal{A}) = \{a \in \mathcal{A} \mid \exists a^{-1}\}$ is open?

$a - c = (1 - c \cdot a^{-1}) \cdot a. \Rightarrow (a-c)^{-1} = a^{-1} \cdot (1 - c \cdot a^{-1})^{-1} = a^{-1} \cdot \sum_{n=0}^{\infty} (c \cdot a^{-1})^n$

the series being absolutely convergent if $\|c \cdot a^{-1}\| \leq \|c\| \, \|a^{-1}\| < 1$.

$\Rightarrow$ If $a \in \mathcal{A} \setminus$ N.I.E.$(\mathcal{A})$ then $a + \{c \in \mathcal{A} \mid \|c\| < \|a^{-1}\|^{-1}\} \subseteq \mathcal{A} \setminus$ N.I.E.$(\mathcal{A})$. $\square$

As a corollary we obtain Theorem 4.3 (a) since if $\mathcal{J}$ is a proper ideal then $\mathcal{J} \subseteq N.I.E.(\mathcal{A})$ - closed. Hence, $\overline{\mathcal{J}} \subseteq N.I.E.(\mathcal{A}) \subseteq \mathcal{A} \setminus \{1\}$.
On the other hand, $\overline{\mathcal{J}}$ is also ideal (why?).

Proof of part (b). Note first that the correspondence $\chi \mapsto \operatorname{Ker} \chi$ is injective since χ is a linear functional and $\operatorname{Ker} \chi$ together with $\chi(1) = 1$ completely determines χ.

It remains to show that the division algebra $\mathcal{A}/\mathcal{J}$ obtained by a maximal ideal $\mathcal{J}$ coincides always with $\mathbb{C}$.

We shall use the Gelfand-Mazur theorem. To this end we have to show that $\mathcal{A}/\mathcal{J}$ has a structure of Banach algebra. We shall prove the later as a consequence of the fact that $\mathcal{J}$ is a closed ideal (by part (a)).

Lemma 4.5. (Reminder: quotient of Banach spaces
Фактор Банахови пространства)

Let V be a Banach space, $W \subseteq V$ be a closed linear subspace. Then V/W is a Banach space w.r.t. the norm (quotient norm):

$$\| [v] \| = \inf_{w \in W} \| v + w \| , \quad \text{where} \quad [v] \equiv v + W \in V/W .$$

Proof. $\| [v_1 + v_2] \| = \inf\limits_{w \in W} \| v_1 + v_2 + w \| \leq \inf\limits_{w \in W} \| v_1 + w \| + \inf\limits_{w \in W} \| v_2 + w \|$

$\| [v] \| = 0 \iff \exists \{w_n\} \subseteq W, \ w_n \to v \iff v \in \overline{W} = W$.

Completeness: let $\| [v_j - v_k] \| \xrightarrow[j,k \to \infty]{} 0$. $\exists \{v_{n_k}\}$ - subsequence s.t. $\| [v_{n_k} - v_{n_{k+1}}] \| < 2^{-k}$. $\Rightarrow$ we can choose v_{n_k} s.t. $\| v_{n_k} - v_{n_{k+1}} \| < 2^{-k}$ (how?).
$\Rightarrow \{v_{n_k}\}$ - fundamental. $\Rightarrow \exists \lim\limits_{k \to \infty} v_{n_k} = v$. Since $\| [v - v_{n_k}] \| \leq \| v - v_{n_k} \|$ then: $\lim\limits_{k \to \infty} [v_{n_k}] = [v]$. $\Rightarrow \lim\limits_{j \to \infty} [v_j] = [v]$ (why?). $\square$

Lemma 4.6 (Quotient Banach algebra).
Let A be a Banach algebra and J be a closed two-sided ideal. Then A/J is a Banach algebra w.r.t. the quotient norm.

Proof. $\|[a\cdot b]\| = \inf_{c\in J} \|a\cdot b + c\| \leq \inf_{c,d\in J} \| \underbrace{(a+c)\cdot(b+d)}_{ab+(cb+ad+cd)}\|$

$\Rightarrow \|[a\cdot b]\| \leq \inf_{c\in J}\|a+c\| \inf_{d\in J}\|b+d\| = \|[a]\|\,\|[b]\|.$ $\square$

Thus, Theorem 4.3 (b) is also proven. Finally, part (c) follows from the definition of the quotient norm on $A/\mathrm{Ker}\,\chi$: since χ coincides with the canonical projection $A/\mathrm{Ker}\,\chi$ then $\|\chi\| = 1$ $(\|[a]\| \leq \|a\|,\ \|[1]\| = \|1\| = 1)$.
This completes the poof of Theorem 4.3. $\square$

2. Existence of maximal ideals and Zorn's Lemma (Лема на Цорн)

Preparation:
Let X be a partially ordered set / частично наредено множество (or, poset) i.e., X is endowed with a binary relation (бинарна релация) $\preceq$ that is
(i) reflexive (рефлексивна): $x \preceq x$
(ii) antisymmetric (антисиметрична): $x \preceq y$ & $y \preceq x \Rightarrow x = y$
(iii) transitive (транзитивна): $x \preceq y$ & $y \preceq z \Rightarrow x \preceq z$
$(\forall x, y, z \in X)$.

A linearly (or, totally) ordered subset $L \subseteq X$ is such a subset so that: $\forall x, y \in L$ either $x \preceq y$ or $y \preceq x$.

Maximal (максимален) element $x \in X$ is such an element that $x \preceq y \Rightarrow x = y$.

Precaution: Maximal element $\neq$ Maximum (or, the largest element).
т.е. максимален елемент не означава най-голям елемент.
The largest element if it exists is maximal element but there might be several maximal elements so that there is no largest element.

Zorn Lemma. (We shall treat it as an axiom.)
Let X be a nonempty poset s.t. $\forall$ linearly ordered subset $L \subseteq X$ has an upper bound z (i.e. $x \preceq z\ \forall x \in L$). Then $\exists$ maximal element in X.

<u>Proposition 4.7</u> Let $\mathcal{A}$ be a unital associative algebra and $\mathcal{J} \subseteq \mathcal{A}$ be a left/right/two-sided ideal that is proper ($\mathcal{J} \neq \mathcal{A}$). Then $\mathcal{J}$ is contained in a maximal left/right/two-sided ideal.

<u>Proof.</u> We consider the case of left ideals (the other cases are similar). Let $M := \{ I \mid \mathcal{J} \subseteq I \subsetneqq \mathcal{A},\ I\text{- left ideal} \} \neq \emptyset$. M is a poset w.r.t. $\subseteq$. Let us show that M satisfies the conditions of Zorn's Lemma.

Let $L \subseteq M$ be linearly ordered, i.e. if $I_1, I_2 \in L$ then $\mathcal{J} \subseteq I_1 \subseteq I_2 \subsetneqq \mathcal{A}$ or $\mathcal{J} \subseteq I_2 \subseteq I_1 \subsetneqq \mathcal{A}$. Set $\mathcal{J}' := \bigcup L$. Then $\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{A}$. We claim that $\mathcal{J}'$ is a proper left ideal (and hence, $\mathcal{J}' \in M$).

a) Why $\mathcal{J}'$ is a linear space? If $a_1, a_2 \in \mathcal{J}' \Rightarrow a_1 \in I_1,\ a_2 \in I_2$ for some $I_1, I_2 \in L$. Assume for the sake of definiteness (да предположим за определеност) that $I_1 \subseteq I_2$. Then $a_1, a_2 \in I_2$. $\Rightarrow \alpha_1 a_1 + \alpha_2 a_2 \in I_2 \subseteq \mathcal{J}'$.

b) If $a \in \mathcal{A}$, $c \in \mathcal{J}'$ why $a \cdot c \in \mathcal{J}'$? $\exists I \in L$ s.t. $c \in I$. $\Rightarrow a \cdot c \in I \subseteq \mathcal{J}'$.

c) Why $\mathcal{J}'$ is proper? $\forall I \in L: I \subseteq N.I.E(\mathcal{A}) \Rightarrow \mathcal{J}' = \bigcup L \subseteq N.I.E(\mathcal{A})$

Hence, by the Zorn Lemma, $\exists$ maximal element in M. $\square$

<u>Corollary 4.8</u> Let $\mathcal{A}$ be a unital associative algebra, $a \in N.I.E.(\mathcal{A})$. Then $\exists$ maximal left (or, right) ideal $\mathcal{J} \subseteq \mathcal{A}$, s.t. $a \in \mathcal{J}$.

<u>Proof.</u> $\mathcal{A}.a$ is a left ideal. $1 \notin \mathcal{A}.a$ (since $a \in N.I.E.(\mathcal{A})$). $\Rightarrow \mathcal{A}.a$ is a proper left ideal. Apply Prop. 4.7. $\square$

3. The Gelfand map

The idea is to consider the characters χ as points of a space associated with a commutative complex associative algebra and every $a \in \mathcal{A}$ as a function with value $\chi(a)$ at χ.

Set: $X(\mathcal{A}) := X := \{ \chi \mid \chi : \mathcal{A} \to \mathbb{C} \text{ is a character} \}$.

$\forall a \in \mathcal{A},\ \chi \in X : \ \hat{a}(\chi) := \chi(a)$.

Then $a \mapsto \hat{a}$ is a morphism $\mathcal{A} \to \mathrm{Maps}(X, \mathbb{C})$:

$$\widehat{a \cdot b}(\chi) = \chi(a \cdot b) = \chi(a) \cdot \chi(b) = \hat{a}(\chi)\, \hat{b}(\chi) = (\hat{a} \cdot \hat{b})(\chi).$$

Theorem 4.9. Let $\mathcal{A}$ be a unital commutative Banach algebra. Then for $\forall a \in \mathcal{A}$: $\hat{a}(X) = \sigma_{\mathcal{A}}(a)$.

Proof. $z \in \sigma_{\mathcal{A}}(a) \overset{\text{def}}{\Longleftrightarrow} a - z.1 \in N.I.E.(\mathcal{A})$

$$\overset{\text{Cor. 4.8, Thm. 4.3 (b)}}{\underset{\text{Observation 4.4 (a)}}{\Longleftrightarrow}} \quad \exists \chi \in X: \ a - z.1 \in \operatorname{Ker} \chi \Leftrightarrow \chi(a - z.1) = 0 \Leftrightarrow \chi(a) = \hat{a}(\chi) = z,$$

i.e. $z \in \sigma_{\mathcal{A}}(a) \Longleftrightarrow z \in \hat{a}(X)$. $\square$

Corollary 4.10. Let $\mathcal{A}$ be a unital commutative Banach algebra. Then the Gelfand map $a \mapsto \hat{a}$ is a morphism:

$$\mathcal{A} \to \text{Bounded Maps}(X, \mathbb{C}).$$

Proof. Since $\hat{a}(X) = \sigma_{\mathcal{A}}(a)$ - compact set. $\square$

Corollary 4.11 Every character χ on a unital commutative C^*-algebra $\mathcal{A}$ is real, i.e. $\chi(a^*) = \overline{\chi(a)}$.

Proof. Enough to prove that $a = a^* \Rightarrow \chi(a) \in \mathbb{R}$.
But if $a = a^*$ then $\mathbb{R} \supseteq \sigma_{\mathcal{A}}(a) = \hat{a}(X) \ni \hat{a}(\chi) = \chi(a)$. $\square$

Corollary 4.12 For a unital commutative C^*-algebra $\mathcal{A}$ the Gelfand map $\mathcal{A} \to \text{Bounded Maps}(X, \mathbb{C})$ is an isometric injection.

Proof. Recall from Lecture 2: $\|a\|^2 = \rho_{\mathcal{A}}(a^*a) =$

$$= \sup\{|z| \mid z \in \sigma_{\mathcal{A}}(a^*a) = \widehat{a^*a}(X)\} = \sup_{\chi \in X} \widehat{a^*a}(\chi) = \sup_{\chi \in X} \chi(a^*a)$$

$$= \left(\sup_{\chi \in X} |\chi(a)|\right)^2 = \left(\sup_{\chi \in X} |\hat{a}(\chi)|\right)^2 = \|\hat{a}\|^2.$$

In particular, $a \mapsto \hat{a}$ is injection. $\square$

4. Injectivity of the Gelfand map for Banach algebras and the Jacobson radical

Def. The Jacobson radical in an associative complex algebra $\mathcal{A}$ is called the intersection of all maximal two-side ideals of $\mathcal{A}$. It is again a two-sided ideal.

Observation. Let $\mathcal{A}$ be a (unital) commutative Banach algebra. Then the kernel of the Gelfand map is the Jacobson radical

Proof. $\hat{a} = 0 \iff \forall \chi \in X : 0 = \hat{a}(\chi) = \chi(a) \iff \forall \chi \in X : a \in \operatorname{Ker} \chi$.
By Thm. 4.3 (b): $\hat{a} = 0 \iff a \in \bigcap \{\text{all maximal ideals}\}$. $\square$

5. Commutative C^*-algebras generated by one element.

Proposition 4.13. Let $\mathcal{A}$ be a unital commutative C^*-algebra generated by an element $a = a^*$. Then $\mathcal{A} \cong C(\sigma_{\mathcal{A}}(a))$.
(the algebra of continuous functions on $\sigma_{\mathcal{A}}(a)$).

Proof. Introduce two subalgebras:

$\mathbb{C}[a] = \{ p(a) \mid p(t) \in \mathbb{C}[t] \text{ - the polynomial algebra} \}$

$\mathbb{C}[t]|_{\sigma_{\mathcal{A}}(a)} := \{ f \in C(\sigma_{\mathcal{A}}(a)) \mid \exists p \in \mathbb{C}[t] \; \forall t \in \sigma_{\mathcal{A}}(a) : f(t) = p(t) \}$

The we note that for $\forall p(t) \in \mathbb{C}[t]$:

$$\widehat{p(a)} = \left(p|_{\sigma_{\mathcal{A}}(a)}\right) \circ \hat{a}$$

$$\begin{tikzcd} & \mathbb{C} & \\ X \arrow[ur, "\widehat{p(a)}"] \arrow[rr, two heads, "\hat{a}"'] & & \sigma_{\mathcal{A}}(a) \arrow[ul, "p|_{\sigma_{\mathcal{A}}(a)}"'] \end{tikzcd}$$

Indeed: for $p(t) = t^n$, $\widehat{p(a)} = \widehat{a^n} = (\hat{a})^n = p(\hat{a}) = p \circ \hat{a}$. Since $\hat{a} : X \twoheadrightarrow \sigma_{\mathcal{A}}(a)$ is surjection then $f = p|_{\sigma_{\mathcal{A}}(a)}$ is the unique solution of $\widehat{p(a)} = f \circ \hat{a}$. Even more: $\sup_{\chi \in X} |\widehat{p(a)}(\chi)| = \sup_{t \in \sigma_{\mathcal{A}}(a)} |p(t)|$.
Hence, $\|p(a)\| = \|p|_{\sigma_{\mathcal{A}}(a)}\|$.

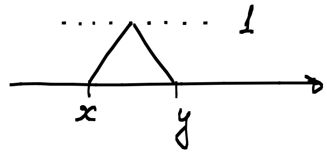
Thus $\mathbb{C}[a] \cong \mathbb{C}[t]|_{\sigma_{\mathcal{A}}(a)}$ are two isometrically isomorphic subalgebras in $\mathcal{A}$ and $C(\sigma_{\mathcal{A}}(a))$, respectively.

By assumption, $\mathbb{C}[a]$ is dense in $\mathcal{A}$ (this means that $\mathcal{A}$ is generated by a as a Banach algebra).

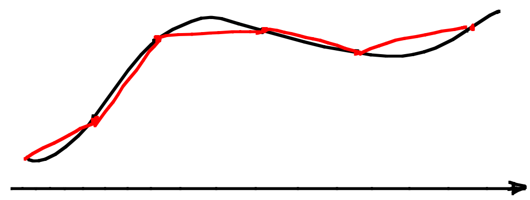
It follows then that $\mathcal{A} \cong$ uniform closure of $\mathbb{C}[t]|_{\sigma_{\mathcal{A}}(a)}$.
This closure coincides with $C(\sigma_{\mathcal{A}}(a))$ by the Weierstrass theorem.
(Теорема на Ваерщрас) □

<u>Weierstrass theorem</u> $\mathbb{C}[t]|_{[\alpha,\beta]}$ is uniformly dense in $C[\alpha,\beta]$. □

The idea of the proof of Weierstrass theorem is first to prove that every function of a form



can be uniformly approximated by polynomials; then using such functions one approximates uniformly every continuous function by piece-wise linear (засоитко линейни) functions



<u>Remarks</u> Note that $\sigma_{\mathcal{A}}(a) \subseteq [-\rho_{\mathcal{A}}(a), \rho_{\mathcal{A}}(a)]$ and
$$\|p(t)\|_{\sigma_{\mathcal{A}}(a)} \leq \|p(t)\|_{[-\rho_{\mathcal{A}}(a),\, \rho_{\mathcal{A}}(a)]} \quad (\forall p \in \mathbb{C}[t]).$$

Thus, every unital commutative C^*-algebra $\mathcal{A}$ is canonically isometrically embedded:
$$\mathcal{A} \hookrightarrow \text{Bounded Maps}(X(\mathcal{A}), \mathbb{C}) : a \mapsto \hat{a}.$$
We shall characterize the image of this algebra by a Hausdorff (Хаусдорф) compact topology on $X(\mathcal{A})$ s.t. $\mathcal{A} \cong C(X(\mathcal{A}))$.

Prop. 4.13 completes the functional calculus in C^*-algebras:
For $a \in \mathcal{A}$ – a unital commutative C^*-algebra, $a = a^*$ (hermitian) if $\mathcal{A}_0$ is the C^*-subalgebra generated by a (and 1) then $\mathcal{A}_0 \cong C(\sigma_{\mathcal{A}}(a))$ and for every continuous $F : \sigma_{\mathcal{A}}(a) \to \mathbb{C}$ we define $F(a) \longleftrightarrow \widehat{F(a)} := F \circ \hat{a}$.

Still, to have a complete spectral decomposition for a we need to construct $F(a)$ for characteristic functions but they do not belong, in general, to $\sigma_{\mathcal{A}}(a)$ (if $\sigma_{\mathcal{A}}(a)$ is connected, for instance.

We proceed to the topological part of the Gelfand theory.
Let us formulate the final result:

Gelfand Theorem

(a) Let $\mathcal{A}$ be a unital commutative C^*-algebra. Then there exists an isometrical isomorphism $\mathcal{A} \cong C(Y)$, where Y is a compact Hausdorff (Хаусдорф) space and $C(Y)$ is the algebra of continuous functions on Y.

(b) For every compact Hausdorff space Y there is a canonical isomorphism:

$$Y \xrightarrow{\simeq} X(C(Y)) \left(= \begin{array}{l}\text{the set of characters} \\ \text{of } C(Y)\end{array} \right)$$

$$y \longmapsto \chi_y, \quad \chi_y(f) := f(y).$$

(c) For every Y, Z - compact Hausdorff spaces and $X \xrightarrow{F} Y$ - a continuous map, the induced map: $F^*: C(Z) \to C(Y)$ where $F^*(f) := F \circ f$ is a morphism of unital C^*-algebras.

$$\begin{tikzcd} & \mathbb{C} & \\ Y \arrow[ur, "F^*(f)"] \arrow[rr, "F"'] & & Z \arrow[ul, "f"'] \end{tikzcd}$$

(d) Every (algebraic) morphism $\varphi: C(Z) \to C(Y)$ of unital $*$-algebras is of a form $\varphi = F^*$ for a unique $F: Y \to Z$ a continuous map between the compact Hausdorff spaces Y and Z.

6. Basic facts from topology.

1) Nations from set theory

X - set, $P(X)$ - the set of all subsets of X

If $\mathcal{F} \subseteq P(X)$ then $\cup\mathcal{F} := \bigcup_{Y \in \mathcal{F}} Y$, $\cap \mathcal{F} := \bigcap_{Y \in \mathcal{F}} Y$

Also, if: $\mathcal{F} = \{Y_\alpha : \alpha \in A\}$, then $\cup\mathcal{F} \equiv \bigcup_\alpha Y_\alpha$, $\cap\mathcal{F} = \bigcap_\alpha Y_\alpha$.

2) Topology on X is $\mathcal{T} \subseteq P(X)$ s.t:

(O1) $\phi \in \mathcal{T}$, $X \in \mathcal{T}$

(O2) If $\mathcal{F} \subseteq \mathcal{T}$ then $\cup\mathcal{F} \in \mathcal{T}$

(O3) If $\mathcal{F} \underset{\text{finite}}{\subseteq} \mathcal{T}$ then $\cap\mathcal{F} \in \mathcal{T}$

$\mathcal{U} \subseteq X$ - open (in X) $\underset{\text{def}}{\Leftrightarrow}$ $\mathcal{U} \in \mathcal{T}$

$\mathcal{U}$ is a neighbourhood (околност) of $x \in X$ $\underset{\text{def}}{\Leftrightarrow}$ $\mathcal{U}$ is open and $x \in \mathcal{U}$.

$\mathcal{U}$ is closed (in X) $\underset{\text{def}}{\Leftrightarrow}$ $X \setminus \mathcal{U}$ is open

<u>Prop.</u> $\mathcal{U}$ is open $\Leftrightarrow$ $\forall$ point of $\mathcal{U}$ lies in $\mathcal{U}$ together with a neighbourhood

3) $\{\phi, X\}$ - indiscrete topology (антидискретна топология) of X

$P(X)$ - discrete topology (дискретна топология) of X
(every point is open and closed subset)

Topological space with discrete topology = discrete space

X is a discrete space $\underset{\text{exercise}}{\Leftrightarrow}$ $\forall x \in X$, $\{x\}$ is open set
(every point is open set)

4) $(X, \mathcal{T})$, $(X', \mathcal{T}')$ - topological spaces

$X \xrightarrow{f} X'$ - continuous $\underset{\text{def}}{\Leftrightarrow}$ $f^{-1}(\mathcal{U}') \in \mathcal{T}$ $(\forall \mathcal{U}' \in \mathcal{T}')$.

<u>Prop.</u> f is continuous $\Leftrightarrow$ $\forall x' \in X'$, $x' = f(x)$, $\forall$ neighbourhood $\mathcal{U}'_{x'}$
$\exists$ neighbourhood $\mathcal{U}_x$ s.t. $f(\mathcal{U}_x) \subseteq \mathcal{U}'_{x'}$.

5.) Convergent sequence $x_n \to x$: $\forall U$ - neighbourhood of x, $\exists n$ s.t.: $x_m \in U$ $\forall m > n$.

6.) The poset of all topologies on X

$\mathcal{T}_1 \subseteq \mathcal{T}_2$

finer, stronger (по-фина, по-силна)

coarser, weaker (по-груба, по-слаба)

$\mathcal{T} \uparrow$

$\{$continuous functions $X \to \mathbb{R}\} \uparrow$

$\{$convergent sequences$\} \downarrow$

Indiscrete topology - the most coarse topology.
continuous functions - only constant functions,
convergent sequences - all sequences.

Discrete topology - the most fine topology.
continuous functions - all functions
convergent sequences - only almost constant sequences.

7.) $\mathcal{T}_\alpha$ - collection of topologies of X, then $\bigcap_\alpha \mathcal{T}_\alpha$ is also topology - the most fine topology coarser than $\forall \mathcal{T}_\alpha$.

$\bigcap \{\mathcal{T}'$ - топ.: $\mathcal{T}_\alpha \subseteq \mathcal{T}' \ (\forall \alpha)\}$ - the weakest topology, that is stronger than $\forall \mathcal{T}_\alpha$. $=: \bigvee_\alpha \mathcal{T}_\alpha$

8) $(X, \mathcal{T})$ - topological space, $Y \subseteq X$ - subspace

$\mathcal{T}' := \{ U \cap Y : U \in \mathcal{T} \}$ - topology in Y

Note: if $A \subseteq Y \subseteq X$ then

A is open in Y $\not\Leftrightarrow$ A is open in X !

A is closed in Y $\not\Leftrightarrow$ A is closed in X !

But: if Y is open in X then A is open in Y $\Leftrightarrow$ A is open in X

Y is closed in X then A is closed in X $\Leftrightarrow$ A is closed in X.

<u>Опр.</u> Едно подмножество $S \subseteq X$ в топ. пр-во X се нарича дискретно, ако индуцираната топология в S е дискретната топология.

Упр. S е дискретно $\Longleftrightarrow$ $\forall x \in S\ \exists U_x \ni x$ - околност, т.че $U_x \cap S = \{x\}$ (всяка точка на S може да се отдели с околност от останалите).

Hausdorff

12.) Хаусдорфови пр-ва: $\forall x \neq y$, $(x, y \in X)$ $\exists U_x (\ni x)$, $U_y (\ni y)$ - околности т.че $U_x \cap U_y = \emptyset$.

Пример: дискретната топология е хаусдорфова
антидискретната - не, ако $|X| \geq 2$.

13.) Компактна топология (компактни пр-ва):

Опр. Отворено покритие на X: $\mathcal{F} \subseteq \mathcal{T}$ т.че $\bigcup \mathcal{F} = X$

open cover has a finite subcover

Компактност $\overset{}{\underset{def}{\Longleftrightarrow}}$ $\forall$ отворено покритие $\mathcal{F}$ притежава крайно подпокритие $(\exists \mathcal{F}_0 \underset{\text{крайно}}{\subseteq} \mathcal{F}$ т.че $\bigcup \mathcal{F}_0 = X)$

Компактни подмножество Y $\underset{def}{\Longleftrightarrow}$ Y е компактно спрямо топологията на подпространство (т.е. индуцираната).

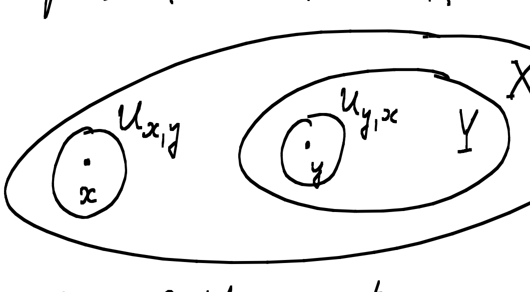
$\underset{\text{Тв.}}{\Longleftrightarrow}$ $\forall$ отворено покритие на Y ($\mathcal{F} \subseteq \mathcal{T}$ т.че $\bigcup \mathcal{F} \supseteq Y$) притежава крайно подпокритие ($\mathcal{F}_0 \underset{\text{кр.}}{\subseteq} \mathcal{F}$ т.че $\bigcup \mathcal{F}_0 \supseteq Y$).

Теорема В компактно пр-во X:

$$Y \text{ е затворено } \underset{\text{ако } X \text{ е хаусдорфово}}{\overset{\longrightarrow}{\longleftarrow}} Y \text{ е компактно}$$

Док: $\Rightarrow$) Ако $\mathcal{F}$ - отв. покритие на Y, то $\mathcal{F} \cup \{X \setminus Y\}$ - отв. покр. на X
$\Rightarrow$ $\mathcal{F}_0 \cup \{X \setminus Y\}$ - крайно подпокр. на X $\Rightarrow$ $\mathcal{F}_0$ е подпокр. на Y.

$\Leftarrow$) Фиксираме произволно $x \in X \setminus Y$ - целим да покажем, че x е вътрешна точка на $X \setminus Y$, т.е. че $\exists U_x \ni x$ - околност, $U_x \subseteq X \setminus Y$.



$\underset{\ni x}{U_{x,y}} \cap \underset{\ni y}{U_{y,x}} = \emptyset$

- хаусдорфовост

$\{U_{y,x}\}_{y \in Y}$ - отв. покритие на Y

$\Rightarrow Y \subseteq U_{y_1,x} \cup \dots \cup U_{y_n,x}$

$\Rightarrow U_{x,y_1} \cap \dots \cap U_{x,y_n}$ - околност на x дизюнктна на Y ($U_x \cap Y = \emptyset$).

Теорема на Вайерщрас: $f: X \to Y$ – непрекъсната

$(X, \mathcal{T})$ – компактно $\Rightarrow$ $f(X)$ е компактно.

Док: Ако $\mathcal{F}$ е отв. покритие на $f(X)$, то $\{f^{-1}(U) : U \in \mathcal{F}\}$ е отв. покритие на X $\Rightarrow$ $\bigcup_{k=1}^{n} f^{-1}(U_k) = X$ $\Rightarrow$ $\bigcup_{k=1}^{n} U_k \supseteq f(X)$. $\square$

Следствие: Хаусдорфовите компактни топологии са минимални топологии в множеството на хаусдорфовите топологии и обратното.

Proof. $(X, \mathcal{T}) \xrightarrow{id} (X, \mathcal{T}')$ is continuous $\Longleftrightarrow$ $\mathcal{T}' \subseteq \mathcal{T}$.

Then if $\mathcal{T}$ is compact and Hausdorff $id(F) = F$ – compact in $\mathcal{T}'$ for $\forall F$ – compact in $\mathcal{T}$. $\Rightarrow$ $\forall$ $\mathcal{T}$-closed is $\mathcal{T}'$-closed set.

$\Rightarrow \mathcal{T} = \mathcal{T}'$. $\square$

Heine - Borel

Теорема (на Хайне - Борел).

В $\mathbb{R}$: $K \subseteq \mathbb{R}$ е компактно $\Longleftrightarrow$ затворено и ограничено

Следствие: (Класическа теорема на Вайерщрас):

$f: \underset{\text{комп.}}{X} \to \mathbb{R}$ $\Rightarrow$ f е ограничена и f достига най-голямата и най-малката си стойности

$\Uparrow$ $f(X)$ е ограничено $\quad$ $\Uparrow$ $f(X)$ е затворено